

CHARACTER SUMS FOR PRIMITIVE ROOT DENSITIES

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ABSTRACT. It follows from the work of Artin and Hooley that, under assumption of the generalized Riemann hypothesis, the density of the set of primes q for which a given non-zero rational number r is a primitive root modulo q can be written as an infinite product $\prod_p \delta_p$ of local factors δ_p reflecting the degree of the splitting field of $X^p - r$ at the primes p , multiplied by a somewhat complicated factor that corrects for the ‘entanglement’ of these splitting fields.

We show how the correction factors arising in Artin’s original primitive root problem and some of its generalizations can be interpreted as character sums describing the nature of the entanglement. The resulting description in terms of local contributions is so transparent that it greatly facilitates explicit computations, and naturally leads to non-vanishing criteria for the correction factors.

1. INTRODUCTION

Artin’s conjecture on primitive roots, which dates back to 1927, states that for a non-zero rational number $r \neq \pm 1$, the set of primes q with the property that r is a primitive root modulo q has natural density

$$\delta(r) = c_r \cdot \prod_{p \text{ prime}} \left(1 - \frac{1}{p(p-1)}\right)$$

inside the set of all primes. Here p ranges over the rational primes, and c_r is a rational number that depends on r . The universal constant $\prod_p (1 - \frac{1}{p(p-1)}) \doteq .3739558$ is known as *Artin’s constant*. The constant c_r vanishes if and only if r is a square. For values of r that are not perfect powers, it was discovered after computer calculations in 1957 that c_r can be different from 1, leading to a correction of the original conjecture by Artin himself [13]. In 1967, this corrected conjecture was proved under assumption of the generalized Riemann hypothesis (GRH) by Hooley [2].

The heuristic argument underlying Artin’s conjecture is simple: for a prime number $q \nmid r$, the number r is a primitive root modulo q if and only if there is no prime number p dividing $q - 1$ such that r is a p -th power modulo q . In terms of number

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fields, this condition amounts to saying that for no prime $p < q$, the prime q splits completely in the splitting field

$$(1.1) \quad F_p = \mathbf{Q}(\zeta_p, \sqrt[p]{r}) \subset \overline{\mathbf{Q}}$$

of the polynomial $X^p - r$ over \mathbf{Q} . Here $\overline{\mathbf{Q}}$ denotes an algebraic closure of \mathbf{Q} , and ζ_p a primitive p -th root of unity in $\overline{\mathbf{Q}}$.

For fixed p , the set of primes q that do not split completely in F_p has density $\delta_p = 1 - [F_p : \mathbf{Q}]^{-1}$. As we have $r \neq \pm 1$, there is a largest integer $h \in \mathbf{Z}$ for which r is an h -th power in \mathbf{Q}^* . We have $[F_p : \mathbf{Q}] = p - 1$ for p dividing h , and $[F_p : \mathbf{Q}] = p(p - 1)$ otherwise. If we assume that the splitting conditions at the various primes p that we impose on q are ‘independent’, it seems reasonable to conjecture that $\delta(r)$ equals

$$(1.2) \quad A(r) = \prod_p \delta_p = \prod_p \left(1 - \frac{1}{[F_p : \mathbf{Q}]}\right) = \prod_{p|h} \left(1 - \frac{1}{p-1}\right) \cdot \prod_{p \nmid h} \left(1 - \frac{1}{p(p-1)}\right).$$

Note that $A(r)$ is a rational multiple of Artin’s constant, and equal to it for $h = 1$. We have $A(r) = 0$ if and only if r is a square; in this case we have $\delta_2 = 0$, and r is not a primitive root modulo any odd prime q .

The assumption on the independence of the splitting conditions in the various fields F_p is not always correct. If $F_2 = \mathbf{Q}(\sqrt{r})$ is a quadratic field of discriminant D , then it is contained in the $|D|$ -th cyclotomic field $\mathbf{Q}(\zeta_{|D|})$. Thus, if D is *odd*, then F_2 is contained in the compositum of the fields F_p with $p|D$. This is however the only ‘entanglement’ between the fields F_p that occurs. In order to take it into account, one writes $F_n = \mathbf{Q}(\zeta_n, \sqrt[n]{r})$ for the splitting field of $X^n - r$ and applies a standard inclusion-exclusion argument to obtain a conjectural value

$$(1.3) \quad \delta(r) = \sum_{n=1}^{\infty} \frac{\mu(n)}{[F_n : \mathbf{Q}]},$$

where μ denotes the Möbius function. If $F_2 = \mathbf{Q}(\sqrt{r})$ is not quadratic of odd discriminant, then $[F_n : \mathbf{Q}]$ is a multiplicative function on squarefree values of n , and (1.3) reduces to (1.2). If F_2 is quadratic of odd discriminant D , then $[F_n : \mathbf{Q}]$ is no longer multiplicative, as it equals $\frac{1}{2} \prod_{p|n} [F_p : \mathbf{Q}]$ for all squarefree n that are divisible by D . In this case, it is not so clear whether the right hand side of (1.3) is non-zero, or even non-negative. However, a ‘rather harder’ calculation [2, p. 219] shows that it can be written as $\delta(r) = E(r) \cdot A(r)$, with $A(r)$ as in (1.2) and

$$(1.4) \quad E(r) = 1 - \prod_{\substack{p|D \\ p|h}} \frac{-1}{p-2} \cdot \prod_{\substack{p|D \\ p \nmid h}} \frac{-1}{p^2 - p - 1}$$

an ‘entanglement correction factor’. Note that $E(r)$ is well-defined as D is odd. The multiplicative structure of the second term of $E(r)$ immediately shows that $E(r)$ is non-zero.

The explicit form of Artin’s conjecture as we have just stated it, is the version that was proved by Hooley under the assumption of the generalized Riemann hypothesis. The hypothesis is used to obtain sufficient control of the error terms occurring in density statements for sets of primes that split completely in the fields F_n in order to prove (1.3). So far, unconditional results have remained insufficient to handle conditions at infinitely many primes p .

Artin’s conjecture can be generalized in various ways. For example, one may impose the additional condition on the primes q that they lie in a given arithmetic progression. Alternatively, one can replace the condition that r be a primitive root modulo q by the requirement that r generate a subgroup of given index in \mathbf{F}_q^* , or even combine the two conditions. Just as in the original conjecture, these conditions amount to imposing restrictions on the splitting behavior of q in number fields contained in the infinite extension

$$F_\infty = \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} \mathbf{Q}(\zeta_n, \sqrt[n]{r}) \subset \overline{\mathbf{Q}}$$

of \mathbf{Q} . They may be formulated as conditions on the Frobenius element of q in these number fields, for which density statements hold by the Chebotarev density theorem. As was shown by the first author [3], the prime densities for such generalizations can in principle (under assumption of GRH) be obtained along the lines of Hooley’s proof, and equal the ‘fraction’ of good Frobenius elements in G . However, the explicit evaluation of the entanglement correction factor from a formula analogous to (1.3) rapidly becomes very unpleasant.

The present paper, which was already announced in [13], introduces a simple but effective method to compute entanglement correction factors for primitive root problems over \mathbf{Q} . It starts with the observation that the Galois automorphisms of the field F_∞ act as group automorphisms on the subgroup

$$R_\infty = \{x \in \overline{\mathbf{Q}}^* : x^k \in \langle r \rangle \subset \mathbf{Q}^* \text{ for some } k \in \mathbf{Z}_{>0}\}$$

of $\overline{\mathbf{Q}}^*$ consisting of the *radicals* that generate the infinite field extension F_∞ of \mathbf{Q} .

In Section 2, we show that for *all* $r \in \mathbf{Q}^* \setminus \{\pm 1\}$, this action gives rise to an injective ‘Galois representation’

$$(1.5) \quad G = \text{Gal}(F_\infty/\mathbf{Q}) \longrightarrow A = \text{Aut}_{R_\infty \cap \mathbf{Q}^*}(R_\infty)$$

that embeds G as an open subgroup of index 2 in the group A of group automorphisms of R_∞ that restrict to the identity on $R_\infty \cap \mathbf{Q}^*$. Unlike the Galois group G ,

the automorphism group A is *always* a product of local factors A_p at the primes p . In Theorem 2.7, we explicitly describe the quadratic character $\chi : A \rightarrow \{\pm 1\}$ that has G as its kernel: it is a finite product $\chi = \prod_p \chi_p$ of quadratic characters χ_p , with each χ_p factoring via the projection $A \rightarrow A_p$.

The profinite group $A = \prod_p A_p$ carries a Haar measure ν , which we can take equal to the product $\prod_p \nu_p$ of the normalized Haar measures ν_p on A_p . For *any* subset $S \subset A$ of the form $\prod_p S_p$ with $S_p \subset A_p$ measurable, one can compute the fraction $\delta(S) = \nu(G \cap S)/\nu(G)$ of elements in G that lie in S as a character sum in terms of the quadratic character χ . In our applications, $S \cap G$ will be a set of ‘good’ Frobenius elements inside G . By Hooley’s method, the fraction $\delta(S)$ is then, under GRH, the density of the set of primes q meeting the Frobenius conditions imposed by the choice of S .

In Theorem 3.3, we show that for the sets $S = \prod_p S_p$ as above, the fraction $\delta(S)$ is the natural product of a naive ‘Artin constant’

$$\nu(S) = \frac{\nu(S)}{\nu(A)} = \prod_p \frac{\nu_p(S_p)}{\nu_p(A_p)} = \prod_p \nu_p(S_p)$$

as we gave in (1.2) and an entanglement correction factor of the form

$$(1.6) \quad E = 1 + \prod_p E_p.$$

Just as in (1.4), where we have $E_2 = -1$, the local factors E_p are different from 1 only for finitely many ‘critical’ primes p occurring in the finite product $\chi = \prod_p \chi_p$. The factor

$$E_p = \frac{1}{\nu_p(S_p)} \int_{S_p} \chi_p d\nu_p$$

equals the average value of χ_p on S_p . It is easily evaluated in cases where S_p is a set-theoretic difference of subgroups of A_p , and can usually be computed explicitly as the average value of a quadratic character on a finite set.

The transparent structure of the formula obtained makes it easy to decide when the fraction $\delta(S)$ of good Frobenius elements in G vanishes. Vanishing of the Artin constant $\nu(S)/\nu(A)$ means that $S = \prod_p S_p$ is a set of measure 0. In concrete examples, this implies that S is empty, and that there is a prime p for which S_p is empty because the conditions imposed by S cannot be met ‘at p ’. In the original Artin conjecture, this only happens for $p = 2$ in the case that r is a square.

Vanishing of the entanglement correction factor E is a more subtle phenomenon that does not occur in the original conjecture. In accordance with Theorem 4.1 in [3], it means that there is an incompatibility ‘at a finite level’ between the conditions at the critical primes. Since all E_p , being average values of characters, are bounded

in absolute value by 1, it is easy (cf. Corollary 3.4) to spot the occurrences of $E = 0$ in (1.6). We illustrate this by computing the value $\delta(S)$ and its vanishing criteria in the case of Artin's conjecture (Section 4) and its generalizations to primes in arithmetic progressions (Section 5) and near-primitive roots (Section 6) mentioned above. Our final Section 7 discusses some of the many possible extensions of our method to Artin-like problems of various kinds. It shows that the underlying idea of our method has a wide range of application.

2. THE RADICAL EXTENSION F_∞

Let $r \in \mathbf{Q}^*$ be a non-zero rational number different from ± 1 . Then for every $n \in \mathbf{Z}_{\geq 1}$, the number field $F_n = \mathbf{Q}(R_n) = \mathbf{Q}(\zeta_n, \sqrt[n]{r})$ is obtained by adjoining to \mathbf{Q} a group of *radicals*

$$R_n = \{x \in \overline{\mathbf{Q}}^* : x^n \in \langle r \rangle \subset \mathbf{Q}^*\} \subset \overline{\mathbf{Q}}^*.$$

As R_n is stable under the action of Galois, the action of a *field* automorphism on F_n is completely determined by its action as a *group* automorphism on the group of radicals R_n . This gives rise to a natural injection of finite groups

$$(2.1) \quad \text{Gal}(F_n/\mathbf{Q}) \longrightarrow A(n) = \text{Aut}_{R_n \cap \mathbf{Q}^*}(R_n).$$

The union $R_\infty = \bigcup_{n \geq 1} R_n$ of all radical groups generates an infinite degree extension $F_\infty = \mathbf{Q}(R_\infty)$, and we may take projective limits on both sides of the map above to obtain the injective group homomorphism

$$(1.5) \quad G = \text{Gal}(F_\infty/\mathbf{Q}) \longrightarrow A = \text{Aut}_{R_\infty \cap \mathbf{Q}^*}(R_\infty)$$

from the Introduction. Note that the profinite groups G and A each come equipped with their Krull topology, and that the injection 1.5 is an injection of *topological* groups.

We let $e \in \mathbf{Z}_{>0}$ be the largest integer for which we have $r = \pm r_0^e$, with $r_0 \in \mathbf{Q}_{>0}$. We then have

$$R_\infty \cap \mathbf{Q}^* = \langle r_0 \rangle \times \langle -1 \rangle.$$

The group R_∞ contains the group $\mu_\infty = \bigcup_{n \geq 1} \mu_n(\overline{\mathbf{Q}})$ of roots of unity as a subgroup. If we choose $\overline{\mathbf{Q}}$ as a subfield of \mathbf{C} , then r_0 is an element in the \mathbf{Q} -vector space of positive real numbers. Writing $r_0^{\mathbf{Q}}$ for the 1-dimensional subspace it generates, we have

$$(2.2) \quad R_\infty = r_0^{\mathbf{Q}} \times \mu_\infty.$$

The automorphism group $A = \text{Aut}_{R_\infty \cap \mathbf{Q}^*}(R_\infty)$ comes with a natural restriction map $A \rightarrow \text{Aut}(\mu_\infty)$ that is continuous and surjective, and that admits a continuous

left inverse: extend the action to be the identity on $r_0^{\mathbf{Q}}$. As an automorphism σ of R_∞ that is the identity on μ_∞ is determined by the values $\sigma(r_0^{1/n})/r_0^{1/n} \in \mu_n(\overline{\mathbf{Q}})$ for $n \geq 1$, we see that the lower row in the commutative diagram of topological groups below is a split exact sequence describing A :

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{Gal}(F_\infty/\mathbf{Q}_{\mathrm{ab}}) & \longrightarrow & G & \longrightarrow & \mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/\mathbf{Q}) \longrightarrow 1 \\ & & \downarrow & & \downarrow (1.5) & & \downarrow \wr \\ 1 & \longrightarrow & \mathrm{Hom}(r_0^{\mathbf{Q}}/r_0^{\mathbf{Z}}, \mu_\infty) & \longrightarrow & A & \longrightarrow & \mathrm{Aut}(\mu_\infty) \longrightarrow 1. \end{array}$$

The upper row is exact by Galois theory, and the right vertical isomorphism reflects the fact that the maximal cyclotomic extension $\mathbf{Q}(\mu_\infty)$ of \mathbf{Q} is the maximal abelian extension \mathbf{Q}_{ab} of \mathbf{Q} , which has Galois group $\mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/\mathbf{Q}) = \mathrm{Aut}(\mu_\infty)$. As all automorphisms of μ_∞ are exponentiations, $\mathrm{Aut}(\mu_\infty)$ is isomorphic to the unit group $\widehat{\mathbf{Z}}^*$ of the profinite completion $\widehat{\mathbf{Z}} = \varprojlim_n \mathbf{Z}/n\mathbf{Z}$ of \mathbf{Z} .

We see that, in order to understand G as a subgroup of A , we need to identify the image of the Kummer map

$$\begin{aligned} \mathrm{Gal}(F_\infty/\mathbf{Q}_{\mathrm{ab}}) &\longrightarrow \mathrm{Hom}(r_0^{\mathbf{Q}}/r_0^{\mathbf{Z}}, \mu_\infty) \\ \sigma &\longmapsto [r_0^x \mapsto (r_0^x)^{\sigma-1}]. \end{aligned}$$

By Kummer theory, the image is the abelian group dual to $r_0^{\mathbf{Q}}/(r_0^{\mathbf{Q}} \cap \mathbf{Q}_{\mathrm{ab}}^*)$.

2.3. Lemma. *We have $r_0^{\mathbf{Q}} \cap \mathbf{Q}_{\mathrm{ab}}^* = r_0^{\frac{1}{2}\mathbf{Z}}$, and G is a subgroup of A of index 2.*

Proof. The equality for $r_0^{\mathbf{Q}} \cap \mathbf{Q}_{\mathrm{ab}}^*$ amounts to saying that the largest integer k for which the splitting field of $X^k - r_0$ is abelian over \mathbf{Q} equals 2. As $r_0 \in \mathbf{Q}_{>0}$ is not a perfect power in \mathbf{Q}^* , this is an immediate corollary of Schinzel's theorem on abelian binomials [10, Theorem 2; 11], which implies that this splitting field is abelian if and only if $r_0^{\#\mu_k(\mathbf{Q})}$ is a k -th power in \mathbf{Q}^* .

As $r_0^{\mathbf{Q}}/r_0^{\frac{1}{2}\mathbf{Z}}$ is the quotient of $r_0^{\mathbf{Q}}/r_0^{\mathbf{Z}}$ obtained by dividing out its unique subgroup of order 2 generated by $r_0^{1/2} \bmod r_0^{\mathbf{Z}}$, its μ_∞ -dual is the unique subgroup of index 2 in $\mathrm{Hom}(r_0^{\mathbf{Q}}/r_0^{\mathbf{Z}}, \mu_\infty)$. It follows that G is of index 2 in A as well. \square

Lemma 2.3 also leads to a description of the subgroup G of index 2 in A that arises as the Galois group of F_∞ over \mathbf{Q} . The group $\mathrm{Hom}(r_0^{\mathbf{Q}}/r_0^{\mathbf{Z}}, \mu_\infty) = \mathrm{Hom}(\mathbf{Q}/\mathbf{Z}, \mu_\infty)$ can be viewed as the Tate module

$$(2.4) \quad \widehat{\mu} = \varprojlim_n \mu_n = \mathrm{Hom}(\varprojlim_n (\tfrac{1}{n}\mathbf{Z}/\mathbf{Z}), \mu_\infty) = \mathrm{Hom}(\mathbf{Q}/\mathbf{Z}, \mu_\infty)$$

of the multiplicative group. It is a free module of rank 1 over $\widehat{\mathbf{Z}}$, and the natural action of $\mathrm{Aut}(\mu_\infty) = \widehat{\mathbf{Z}}^*$ on $\mathrm{Hom}(\mathbf{Q}/\mathbf{Z}, \mu_\infty)$ via the second argument is simply the $\widehat{\mathbf{Z}}^*$ -multiplication we have on the $\widehat{\mathbf{Z}}$ -module $\widehat{\mu}$.

From the split exact sequence for A , we see that A is the semidirect product

$$(2.5) \quad A = \text{Hom}(r_0^{\mathbf{Q}}/r_0^{\mathbf{Z}}, \mu_\infty) \rtimes \text{Aut}(\mu_\infty) = \widehat{\mu} \rtimes \widehat{\mathbf{Z}}^*,$$

which is isomorphic to the affine group $\widehat{\mathbf{Z}} \rtimes \widehat{\mathbf{Z}}^*$ over $\widehat{\mathbf{Z}}$.

The subgroup $G \subset A$ is an extension of $\widehat{\mathbf{Z}}^*$ by $\widehat{\mu}^2 \cong \widehat{\mu}$, but this extension is non-split: if it were, the maximal abelian quotient of $G = \text{Gal}(F_\infty/\mathbf{Q})$, which is $\text{Gal}(\mathbf{Q}_{\text{ab}}/\mathbf{Q}) \cong \widehat{\mathbf{Z}}^*$, would be isomorphic to the maximal abelian quotient $\mathbf{Z}/2\mathbf{Z} \times \widehat{\mathbf{Z}}^*$ of the affine group over $\widehat{\mathbf{Z}}$.

By 2.2, the field $F_\infty = \mathbf{Q}(R_\infty)$ is the compositum of $\mathbf{Q}(r_0^{\mathbf{Q}})$ and $\mathbf{Q}(\mu_\infty) = \mathbf{Q}_{\text{ab}}$, and the embedding $G \subset A$, with A in the explicit form 2.5, amounts to a description of the field automorphisms of F_∞ in terms of their action on these constituents. The index 2 of G in A reflects the fact that by Lemma 2.3, the intersection of $\mathbf{Q}(r_0^{\mathbf{Q}})$ and \mathbf{Q}_{ab} is not $\mathbf{Q}(r_0) = \mathbf{Q}$, but the quadratic field $K = \mathbf{Q}(\sqrt{r_0})$. This implies that an element $(\phi, \sigma) \in A$ is in G if and only if ϕ and $\sigma \in \text{Aut}(\mu_\infty) = \text{Gal}(\mathbf{Q}_{\text{ab}}/\mathbf{Q})$ act in a compatible way on $\sqrt{r_0} = r_0^{\frac{1}{2}} \in \mathbf{Q}_{\text{ab}}$:

$$(2.6) \quad \phi(r_0^{\frac{1}{2}}) = (r_0^{\frac{1}{2}})^{\sigma-1} \in \mu_2.$$

In words: an automorphism of the multiplicative group of radicals R_∞ induces an automorphism of the field $F_\infty = \mathbf{Q}(R_\infty)$ if and only if it respects the *additive* identity of $\sqrt{r_0} = r_0^{1/2}$ as a \mathbf{Q} -linear combination of roots of unity with rational coefficients. We can phrase this slightly more formally by saying that $G \subset A$ is the difference kernel of two distinct quadratic characters $\psi_K, \chi_K : A \rightarrow \mu_2$ related to the *entanglement field* $K = \mathbf{Q}(\sqrt{r_0})$.

The first quadratic character $\psi_K : A \rightarrow \{\pm 1\}$ describes the action on the generator $r_0^{1/2}$ of K in terms of the ϕ -component of $a = (\phi, \sigma) \in A$:

$$\psi_K(a) = \phi(r_0^{\frac{1}{2}}) \in \mu_2.$$

Note that ψ_K is indeed a character on A , as $\text{Aut}(\mu_\infty)$ acts trivially on $\mu_2 = \{\pm 1\}$. The second character

$$\chi_K = \left(\frac{r_0}{\cdot} \right) : A \rightarrow \text{Aut}(\mu_\infty) = \widehat{\mathbf{Z}}^* \rightarrow \mu_2$$

is the ‘cyclotomic’ character of conductor $d = \text{disc}(K)$ associated to the entanglement field $K = \mathbf{Q}(\sqrt{r_0})$. This character factors via the quotient $(\mathbf{Z}/d\mathbf{Z})^*$ of the cyclotomic component $\text{Aut}(\mu_\infty) = \widehat{\mathbf{Z}}^*$ of A , on which it can be viewed as the Dirichlet character. Its value in $a \in (\mathbf{Z}/d\mathbf{Z})^*$ is given by the Kronecker symbol $(\frac{r_0}{a})$ corresponding to K .

2.7. Theorem. *Let $K = \mathbf{Q}(\sqrt{r_0})$ and $\psi_K, \chi_K : A \rightarrow \mu_2$ be defined as above. Then the natural map (1.5) identifies $G = \text{Gal}(F_\infty/\mathbf{Q})$ with the subgroup of A of index 2 that arises as the kernel of the quadratic character*

$$\psi_K \cdot \chi_K : A \rightarrow \mu_2.$$

□

From the description in 2.5, or more directly from the fact that automorphisms of R_∞ over $R_\infty \cap \mathbf{Q}^*$ can be given in terms of their action on prime power radicals, it is clear that A admits a natural decomposition

$$(2.8) \quad A = \prod_{p \text{ prime}} A_p,$$

with A_p the group of automorphisms of the group $R_{p^\infty} = \bigcup_{k \geq 1} R_{p^k}$ of p -power radicals that restrict to the identity on $R_{p^\infty} \cap \mathbf{Q}^*$.

The character ψ_K in Theorem 2.7 factors via the component A_2 of A . The other character χ_K can be decomposed in the standard way for Dirichlet characters as a product

$$\chi_K = \prod_p \chi_{K,p}$$

of quadratic characters

$$\chi_{K,p} : A \rightarrow \text{Aut}(\mu_\infty) = \widehat{\mathbf{Z}}^* \rightarrow \mathbf{Z}_p^* \rightarrow \mu_2$$

of p -power conductor that are non-trivial exactly for primes p dividing $d = \text{disc}(K)$. For odd primes $p|d$, the character $\chi_{K,p}$ is a lift to A of the Legendre symbol at p , and $\chi_{K,2}$ is a lift to A of a Dirichlet character of conductor dividing 8. Note that $\chi_{K,p}$ factors via A_p for all p .

2.9. Remark. For odd p , one may identify the p -component A_p of A with the Galois group of $F_{p^\infty} = \mathbf{Q}(R_{p^\infty})$ over \mathbf{Q} . For $p = 2$, this is only true if we are not in the special case where the entanglement field $K = \mathbf{Q}(\sqrt{r_0})$ equals $\mathbf{Q}(\sqrt{2})$. In non-special cases, there is a true ‘entanglement’ of the extensions F_{p^∞} in the sense that the character $\psi_K \cdot \chi_K$ in Theorem 2.7 that determines G as a subgroup of A is non-trivial on more than one prime component A_p . In the special case $K = \mathbf{Q}(\sqrt{2})$, we have $d = 8$ and there is no entanglement at the level of Galois groups; we do however have $G = G_2 \times \prod_{p>2} A_p$ for a subgroup $G_2 \subset A_2$ of index 2.

As we saw in 2.5, the semidirect product $A = \widehat{\mu} \rtimes \widehat{\mathbf{Z}}^*$ is a split extension of $\widehat{\mathbf{Z}}^*$ by $\widehat{\mu}$, but its subgroup $G \subset A$ ‘cut out’ by r_0 in the sense of condition 2.6 is not. It is a *non-split* extension of $\widehat{\mathbf{Z}}^*$ by the subgroup $\widehat{\mu}^2 \subset \widehat{\mu}$ of index 2, which is again

isomorphic to $\widehat{\mu}$. Even though this is not directly relevant for us, one may wonder which non-split extensions

$$\varepsilon_{r_0} : \quad 1 \rightarrow \widehat{\mu} \longrightarrow G \longrightarrow \widehat{\mathbf{Z}}^* \rightarrow 1$$

are provided by the Galois groups $G = \text{Gal}(F_\infty/\mathbf{Q})$ for various choices of r_0 . The apparent answer is that *every* non-split extension of $\widehat{\mathbf{Z}}^*$ by $\widehat{\mu}$ arises in this way, for a unique real quadratic field $\mathbf{Q}(\sqrt{r_0})$. A more formal way to phrase this would be the construction of an isomorphism

$$\mathbf{Q}_{>0}^*/(\mathbf{Q}_{>0}^*)^2 \xrightarrow{\sim} H^2(\widehat{\mathbf{Z}}^*, \widehat{\mu})$$

that maps an element $r_0 \in \mathbf{Q}_{>0}^*/(\mathbf{Q}_{>0}^*)^2$ representing the real quadratic field $\mathbf{Q}(\sqrt{r_0})$ to the class of the extension ε_{r_0} in a continuous cochain cohomology group $H^2(\widehat{\mathbf{Z}}^*, \widehat{\mu})$ that describes profinite group extensions in the spirit of [9, Theorem 6.8.4]. Such a construction can be given by standard arguments, taking $\widehat{\mathbf{Z}}^*$ -cohomology of the sequence $1 \rightarrow \widehat{\mu} \xrightarrow{\square} \widehat{\mu} \longrightarrow \mu_2 \rightarrow 1$ describing multiplication by 2 on $\widehat{\mu}$, as soon as someone establishes the necessary formal properties of continuous cochain cohomology groups $H^q(G, A)$ for profinite rather than simply discrete G -modules A . At this point, such properties do not seem to occur in the literature.

Although we refer to the real quadratic field $K = \mathbf{Q}(\sqrt{r_0})$ in Theorem 2.7 as ‘the’ entanglement field, we should point out that K is only unique up to twisting by the cyclotomic character $\chi_{-4} : A \rightarrow \text{Aut}(\mu_\infty) = \widehat{\mathbf{Z}}^* \rightarrow \{\pm 1\}$ giving the action on $i = \sqrt{-1}$. In other words, G is also the subgroup of A on which the two different characters related to field $K' = \mathbf{Q}(\sqrt{-r_0})$ coincide. Indeed, the product of the two characters $\psi_{K'} = \psi_K \cdot \chi_{-4}$ and $\chi_{K'} = \chi_K \cdot \chi_{-4}$ related to $\sqrt{-r_0}$ yields the same character $\psi_K \cdot \chi_K$ defining G .

The freedom in the choice of the sign of r_0 means that in 2.7, we can take $K = \mathbf{Q}(\sqrt{r_1})$, with $r_1 = \pm r_0$ chosen in such a way that we have

$$(2.10) \quad r = \begin{cases} -r_1^e & \text{if } -r \text{ is a square;} \\ r_1^e & \text{otherwise.} \end{cases}$$

We usually refer to first case, in which r itself is not an e -th power and the field $\mathbf{Q}(\sqrt{r}) = \mathbf{Q}(i)$ is the Gaussian number field, as the *twisted case*. As we will see in Sections 5 and 6, the twisted case often requires special attention in explicit computations.

3. ENTANGLEMENT CORRECTION USING CHARACTER SUMS

The automorphism group A and each of its components A_p in 2.8 are infinite profinite groups that naturally come with a topology and a Haar measure. The

quadratic character $\psi_K \cdot \chi_K$ in Theorem 2.7 is continuous on A with respect to this topology, so G is a closed open subgroup of A . We normalize the Haar measure ν_p on A_p by putting $\nu_p(A_p) = 1$, and this makes the product measure $\nu = \prod_p \nu_p$ into a normalized Haar measure on A .

Densities for Artin-like primitive root problems (in one generator over \mathbf{Q}) arise as fractions $\delta(S) = \nu(G \cap S)/\nu(G)$ of ‘good’ Frobenius elements inside the Galois group $G = \text{Gal}(F_\infty/\mathbf{Q})$ of Theorem 2.7. Here

$$S = \prod_p S_p \subset \prod_p A_p = A$$

is some measurable subset of A that is defined componentwise at each prime p . Usually S_p is the inverse image of some finite set $\overline{S}_p \subset \overline{A}_p$ under a continuous map $A_p \rightarrow \overline{A}_p$ to a finite discrete group \overline{A}_p . A frequently encountered example is, for P a power of p , the restriction map

$$(3.1) \quad \varphi_P : A_p \longrightarrow A(P) = \text{Aut}_{R_P \cap \mathbf{Q}^*}(R_P).$$

Note that unlike R_∞ , the group R_P of all P -th roots of $\langle r \rangle$ depends on r , not just on r_0 . For Artin’s original conjecture, the local condition at p on the Frobenius element is that it is non-trivial on the field $F_p = \mathbf{Q}(R_p) = \mathbf{Q}(\zeta_p, \sqrt[p]{r})$ from (1.1), so we take

$$\varphi_p : A_p \longrightarrow A(p) = \text{Aut}_{R_p \cap \mathbf{Q}^*}(R_p)$$

with $\overline{S}_p = A(p) \setminus \{1\}$, and put

$$(3.2) \quad S_p = A_p \setminus \ker \varphi_p = \varphi_p^{-1}[\overline{S}_p].$$

As φ_p is surjective and $A(p) \cong \text{Gal}(F_p/\mathbf{Q})$ has order $[F_p : \mathbf{Q}]$, the subset $S_p \subset A_p$ has measure $\nu_p(S_p) = 1 - [F_p : \mathbf{Q}]^{-1}$. Thus, $S = \prod_p S_p$ has measure

$$\nu(S) = \prod_p \nu_p(S_p) = \prod_p \left(1 - \frac{1}{[F_p : \mathbf{Q}]}\right)$$

equal to the constant $A(r)$ occurring in (1.2). The entanglement correction factor $E(r)$ in (1.4) is the factor by which $\delta(S) = \nu(G \cap S)/\nu(G)$ is different from $\nu(S) = \nu(S)/\nu(A)$ for the subgroup $G \subset A$ of index 2 described by Theorem 2.7. Such entanglement correction factors can be computed in great generality from the following theorem.

3.3. Theorem. *Let $A = \prod_p A_p$ be as in 2.8, with Haar measure $\nu = \prod_p \nu_p$, and $\chi = \prod_p \chi_p : A \rightarrow \{\pm 1\}$ a non-trivial character obtained from a family of continuous quadratic characters $\chi_p : A_p \rightarrow \{\pm 1\}$, with χ_p trivial for almost all primes p . Then*

for $G = \ker \chi$ and $S = \prod_p S_p$ a product of ν_p -measurable subsets $S_p \subset A_p$ with $\nu_p(S_p) > 0$, we have

$$\delta(S) = \frac{\nu(G \cap S)}{\nu(G)} = \left(1 + \prod_p E_p\right) \cdot \frac{\nu(S)}{\nu(A)},$$

with $E_p = E_p(S) = \frac{1}{\nu_p(S_p)} \int_{S_p} \chi_p d\nu_p$ the average value of χ_p on S_p .

Proof. We assume that $\nu(S) = \prod_p \nu_p(S_p)$ is positive, as the theorem trivially holds for $\nu(S) = 0$. We compute $\nu(G \cap S)$ by integrating the characteristic function $(1 + \chi)/2$ of G over the subset $S \subset A$ with respect to ν . As we have $\nu(G) = \frac{1}{2}\nu(A)$ for non-trivial χ , we obtain

$$\frac{\nu(G \cap S)}{\nu(G)} = \frac{1}{\nu(A)} \int_S (1 + \chi) d\nu = \frac{\nu(S)}{\nu(A)} \cdot \left(1 + \frac{1}{\nu(S)} \int_S \chi d\nu\right).$$

Now $\nu(S)$ equals $\prod_p \nu_p(S_p)$, and the integral of $\chi = \prod_p \chi_p$ over $S = \prod_p S_p$ is the product of the values $\int_{S_p} \chi_p d\nu_p$ for all p . The theorem follows. \square

3.4. Corollary. *Suppose the density $\delta(S)$ in 3.3 vanishes for a set S of non-zero measure. Then there exists a sequence $\{\varepsilon_p\}_p$ of signs $\varepsilon_p \in \{\pm 1\}$, almost all equal to 1, such that we have $\prod_p \varepsilon_p = -1$, and $\chi_p = \varepsilon_p$ almost everywhere on S_p .*

Proof. Suppose we have $\delta(S) = 0$ and $\nu(S) > 0$. Then the product $\prod_p E_p$, which is finite as we have $E_p = 1$ at all p at which χ_p is trivial, equals -1 . As every E_p is the average value of a quadratic character on S_p , it is a real number in $[-1, 1]$. It equals 1 (or -1) if and only if χ_p is ν_p -almost everywhere equal to 1 (or -1) on S_p . Thus, $\prod_p E_p = -1$ occurs exactly under the conditions listed. \square

For the Galois group $G = \text{Gal}(F_\infty/\mathbf{Q}) \subset A$ from Theorem 2.7, we are in the situation of Theorem 3.3 if we take

$$(3.5) \quad \chi_p = \begin{cases} \psi_K \cdot \chi_{K,2} & \text{for } p = 2; \\ \chi_{K,p} & \text{for } p > 2. \end{cases}$$

The characters $\chi_{K,2}$ and ψ_K cannot coincide as only $\chi_{K,2}$ factors via the cyclotomic component $\hat{\mathbf{Z}}^*$ of A in 2.5, so χ_2 is always non-trivial. Note also that χ_2 is unchanged if we replace $K = \mathbf{Q}(\sqrt{r_0})$ by $K' = \mathbf{Q}(\sqrt{-r_0})$, as discussed before 2.10.

3.6. Remark. As we noticed in Remark 2.9, it can happen in the situation of Theorem 3.3 that all χ_p 's but one character χ_q are trivial. In this case we have $G = G_q \times \prod_{p \neq q} A_p$ for some subgroup $G_q \subset A_q$ of index 2, and $G \cap S$ will be the same for all subsets $S_q \subset A_q$ having the same intersection $S'_q = S_q \cap G_q$. The correction factor $1 + \prod_p E_p = 1 + E_q$ does however depend on S_q , not only on S'_q .

This is not a contradiction, since we can write $S_q = S'_q \cup S''_q$ as a disjoint union with $S''_q = S_q \cap (A_q \setminus G_q)$ and observe that the right hand side in Theorem 3.3 equals

$$\begin{aligned} \left(1 + \prod_p E_p\right) \frac{\nu(S)}{\nu(A)} &= \frac{1}{\nu(A)} \left(\nu_q(S_q) + \int_{S_q} \chi_q d\nu_q \right) \prod_{p \neq q} \nu_p(S_p) \\ &= \frac{1}{\nu(A)} \left(\nu_q(S_q) + \nu_q(S'_q) - \nu_q(S''_q) \right) \prod_{p \neq q} \nu_p(S_p) \\ &= \frac{1}{\nu(G)} \nu_q(S'_q) \prod_{p \neq q} \nu_p(S_p), \end{aligned}$$

in accordance with the fact that we have $G \cap S = S'_q \times \prod_{p \neq q} S_p$.

4. ARTIN'S CONJECTURE

Theorems 2.7 and 3.3 reduce the computation of the correction factors occurring in many Artin-like problems to fairly mechanical computations. For Artin's original problem, we already noticed in 3.2 that each subset $S_p \subset A_p$ of 'good' Frobenius elements at p equals $S_p = A_p \setminus \ker \varphi_p$ for the natural map

$$\varphi_p : A_p \rightarrow A(p) = \text{Aut}_{R_p \cap \mathbf{Q}^*}(R_p) \cong \text{Gal}(F_p/\mathbf{Q}),$$

and that this gives rise to an *Artin set* $S = S(r) = \prod_p S_p$ of measure $\nu(S) = A(r)$ as in 1.2. We have $\nu(S) = 0$ if and only if $r \in \mathbf{Q}^* \setminus \{\pm 1\}$ is a square. In this case, S is empty as we have $S_2 = \emptyset$.

To recover the correction factor $E(r)$ from 1.4 for non-square r , we need to compute the entanglement correction factor $1 + \prod_p E_p$ coming out of Theorem 3.3. As $S_p = A_p \setminus \ker \varphi_p$ is the set-theoretic difference of a group and a subgroup, each number

$$E_p = \frac{1}{\nu(S_p)} \left[\int_{A_p} \chi_p d\nu_p - \int_{\ker \varphi_p} \chi_p d\nu_p \right]$$

can only have three possible values, depending on the nature of χ_p . For trivial χ_p , we clearly have $E_p = 1$. If χ_p is non-trivial on $\ker \varphi_p$, and therefore on S_p , we get $E_p = 0$ as both integrals, being integrals of a non-trivial character over a group, vanish. The interesting case is where χ_p is trivial on $\ker \varphi_p$ but not on A_p , since then we obtain

$$(4.1) \quad E_p = \frac{-\nu_p(\ker \varphi_p)}{\nu_p(S_p)} = \frac{-[F_p : \mathbf{Q}]^{-1}}{1 - [F_p : \mathbf{Q}]^{-1}} = \frac{-1}{[F_p : \mathbf{Q}] - 1}.$$

Putting things together, we obtain the following.

4.2. Theorem. *Let $r \neq -1$ be a non-square rational number, $G \subset A$ as in Theorem 2.7, and $S = S(r) \subset A$ the Artin set defined above. Then S has measure $A(r)$ as in 1.2, and we have*

$$\delta(S) = \frac{\nu(G \cap S)}{\nu(G)} = E(r) \cdot A(r)$$

for an entanglement correction factor $E(r)$ that has the value 1 if $D = \text{disc}(\mathbf{Q}(\sqrt{r}))$ is even, and the value

$$E(r) = 1 - \prod_{p|D} \frac{-1}{[F_p : \mathbf{Q}] - 1}$$

from 1.4 if D is odd.

Proof. We can only have a correction factor $E(r) = 1 + \prod_p E_p$ different from 1 if E_2 does not vanish, and that occurs if and only if the non-trivial quadratic character $\chi_2 = \psi_K \cdot \chi_{K,2}$ from 3.5 is *trivial* on the kernel of the quadratic character $\varphi_2 : A_2 \rightarrow \text{Gal}(F_2/\mathbf{Q})$. This means that χ_2 and φ_2 are actually the same character on A_2 , so we are *not* in the twisted case where $-r$ is a square, and have and $K = \mathbf{Q}(\sqrt{r})$ is quadratic of odd discriminant D . We then have $E_2 = -1$ by 4.1, and as the non-trivial characters χ_p at $p|D$ factor via φ_p , the corresponding E_p are also given by 4.1. The value for $E(r)$ follows. \square

The preceding proof is remarkably simple in comparison with the original derivation of (1.4) from (1.3) in [2]. The next two sections show that this character sum analysis generalizes well to more complicated settings.

5. PRIMES IN ARITHMETIC PROGRESSIONS WITH PRESCRIBED PRIMITIVE ROOT

In addition to $r = \pm r_0^e \in \mathbf{Q}^* \setminus \{\pm 1\}$, we choose a *modulus* $f \in \mathbf{Z}_{\geq 1}$ and an integer $a \in \mathbf{Z}$ coprime to f , and write $f = \prod_p f_p$, with $f_p = p^{\text{ord}_p(f)}$ the p -component of f . We are interested in the density of the set of primes

$$\{q \text{ prime} : q \equiv a \pmod{f} \text{ and } r \text{ is a primitive root modulo } q\}.$$

We may and will assume that r is not a square, and that we have $f_2 \neq 2$, i.e., $f \not\equiv 2 \pmod{4}$.

The additional congruence condition $q \equiv a \pmod{f}$ is a condition on the Frobenius of q in the cyclotomic field $\mathbf{Q}(\zeta_f) \subset F_\infty = \mathbf{Q}(R_\infty)$. Using the natural maps

$$\rho_p : A_p = \text{Aut}_{R_{p^\infty} \cap \mathbf{Q}^*}(R_{p^\infty}) \rightarrow \text{Aut}(\mu_{p^\infty}) = \mathbf{Z}_p^* \rightarrow (\mathbf{Z}_p/f\mathbf{Z}_p)^* = (\mathbf{Z}/f\mathbf{Z})^*,$$

we can take it into account by replacing, for each of the finitely many p that divides f , the primitive root set $A_p \setminus \ker \varphi_p$ from 3.2 by its intersection

$$(5.1) \quad S_p = (A_p \setminus \ker \varphi_p) \cap \rho_p^{-1}(a \pmod{f\mathbf{Z}_p})$$

with the congruence set $\rho_p^{-1}(\bar{a}) = \rho_p^{-1}(a \bmod f\mathbf{Z}_p)$. In other words, we map A_p to a finite group by

$$\varphi_p \times \rho_p : A_p \rightarrow \text{Aut}_{R_p \cap \mathbf{Q}^*}(R_p) \times (\mathbf{Z}_p/f\mathbf{Z}_p)^*,$$

and let S_p be the inverse image

$$(\varphi_p \times \rho_p)^{-1} [(\text{Aut}_{R_p \cap \mathbf{Q}^*}(R_p) \setminus \{1\}) \times (a \bmod f\mathbf{Z}_p)].$$

In this way, a prime $q > f$ for which $\text{Frob}_q \in \text{Gal}(\mathbf{Q}(R_{p^\infty})/\mathbf{Q}) \subset A_p$ lies in S_p for all primes $p < q$ will have r as a primitive root *and* lie in the arithmetic progression of integers congruent to $a \bmod f$.

Although each of the maps φ_p and ρ_p is surjective, $\varphi_p \times \rho_p$ often is not, so some care is needed in computing $\nu_p(S_p)$ in 5.1 for the primes p dividing f . We start by observing that S_p is contained in $\rho_p^{-1}(\bar{a})$, a set of measure

$$\nu_p(\rho_p^{-1}(\bar{a})) = (\#(\mathbf{Z}_p/f\mathbf{Z}_p)^*)^{-1} = \phi(f_p)^{-1},$$

with ϕ the Euler ϕ -function.

We now have two cases, depending on $a \bmod p$. For $a \not\equiv 1 \bmod p$, the set $\rho_p^{-1}(\bar{a})$ is disjoint from $\ker \varphi_p$, the congruence condition at p implies the primitive root condition at p , and $S_p = \rho_p^{-1}(\bar{a})$ has measure $\phi(f_p)^{-1}$. For $a \equiv 1 \bmod p$, all elements in $\rho_p^{-1}(\bar{a})$ fix μ_p . If now $r \in \mathbf{Q}^*$ is a p -th power, we have $\text{Aut}_{R_p \cap \mathbf{Q}^*}(R_p) = \text{Aut}(\mu_p)$ and $\rho_p^{-1}(\bar{a})$ is contained in $\ker \varphi_p$. In this case S_p is empty, and r will not be a primitive root modulo any prime $q \equiv 1 \bmod p$. In the more interesting case where r is not a p -th power, an element of $\rho_p^{-1}(\bar{a})$ is in S_p if and only if it does not fix $r^{1/p}$. If we exclude for $p = 2$ dividing f the twisted case, in which $\varphi_2 = \chi_{-4}$ factors via ρ_2 , we find

$$\nu_p(S_p) = (1 - \frac{1}{p})\nu_p(\rho_p^{-1}(\bar{a})) = (1 - \frac{1}{p}) \cdot \phi(f_p)^{-1}$$

whenever p divides $\gcd(a-1, f)$ but not e .

Define the analogue of the ‘naive’ Artin constant $A(r)$ from 1.2 for primes lying in the arithmetic progression $a \bmod f$ by

$$(5.2) \quad A(r, a \bmod f) = \frac{1}{\phi(f)} \prod_{p \mid \gcd(a-1, f)} (1 - \frac{1}{p}) \cdot \prod_{p \nmid f} (1 - \frac{1}{[F_p : \mathbf{Q}]}).$$

Then, under the assumption (necessary for S to be non-empty) that r is not a p -th power for any prime $p \mid \gcd(a-1, f)$, we find that the measure $\nu(S) = \prod_p \nu_p(S_p)$ of the subset $S \subset A$ equals $A(r, a \bmod f)$ whenever we are not in the twisted case with $4 \mid f$.

We now apply Theorem 3.3 to find, in all cases, the density $\delta(S) = \nu(G \cap S)/\nu(G)$ for the Galois group $G \subset A$ from Theorem 2.7. The resulting computation is of striking simplicity when compared to its original derivation by the second author [6, 7] from a formula analogous to 1.3. Under GRH, the fraction $\delta(S)$ equals the density, inside the set of all primes, of the set of primes $q \equiv a \pmod f$ for which r is a primitive root.

5.3. Theorem. *Let $a, f \in \mathbf{Z}_{\geq 1}$ be coprime integers as above, $r \neq -1$ a non-square rational number that is not a p -th power for any prime $p \mid \gcd(a-1, f)$, and define $S = \prod_p S_p \subset A$ associated to the set of primes in the residue class $a \pmod f$ for which r is a primitive root as in 5.1. Then we have*

$$\delta(S) = \frac{\nu(G \cap S)}{\nu(G)} = E \cdot A(r, a \pmod f)$$

for the Galois group $G \subset A$ from 1.5, with $A(r, a \pmod f)$ the Artin constant from 5.2, and the correction factor E equal to

$$E = 1 + E_2 \cdot \prod_{p \mid \gcd(D, f) \text{ odd}} \left(\frac{a}{p} \right) \cdot \prod_{p \mid D, p \nmid 2f} \frac{-1}{[F_p : \mathbf{Q}] - 1}.$$

Here D denotes the discriminant of $F_2 = \mathbf{Q}(\sqrt{r})$, and with $D_2 = 2^{\text{ord}_2(D)}$ we have

$$E_2 = \begin{cases} -\chi_{F_2, 2}(a) & \text{if } D_2 \mid f; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose first that we are *not* in the twisted case $D = -4$. Then the field $K = \mathbf{Q}(\sqrt{r_1})$ defined according to 2.10 is the field $F_2 = \mathbf{Q}(\sqrt{r})$ of discriminant D . Now $\nu(S)/\nu(A)$ equals the constant $A(r, a \pmod f)$ from 5.2, and we have to show that Theorem 3.3, applied for our set S and the characters χ_p from 3.5, leads to a correction factor $E = 1 + \prod_p E_p$ as stated. This amounts to a local computation of E_p at each of the critical primes $p \mid 2D$.

At primes $p \nmid 2f$ dividing D , the factors $E_p = -1/([F_p : \mathbf{Q}] - 1)$ coming from the Legendre symbol χ_p at p are the same as for the original Artin conjecture in 4.1.

For the odd primes $p \mid \gcd(D, f)$, the Legendre symbol χ_p has constant value $\chi_p(a) = \left(\frac{a}{p} \right)$ on the congruence set $\rho_p^{-1}(a \pmod{p^{e_p}})$, and therefore on S_p . This yields $E_p = \left(\frac{a}{p} \right)$ for these p .

For $p = 2$, the character

$$\chi_2 = \psi_K \cdot \chi_{K, 2} = \varphi_2 \cdot \chi_{F_2, 2}$$

equals $-\chi_{F_2, 2}$ on $S_2 \subset A_2 \setminus \ker \varphi_2$. In the case $D_2 \mid f$ it has constant value $-\chi_{F_2, 2}(a)$ on $S_2 \subset \rho_2^{-1}(\bar{a})$, and we obtain $E_2 = -\chi_{F_2, 2}(a)$. In the case $D_2 \nmid f$ we have

$f_2 \in \{1, 4\}$. For $f_2 = 1$ and D even we have $E_2 = 0$ as in the proof of 4.2. For $f_2 = 4$ and $D_2 = 8$ the set S_2 is a coset of the subgroup $\ker \varphi_2 \cap \ker \rho_2 \subset A_2$ of index 4, and we find $E_2 = 0$ as $\chi_{F_2,2}$ is non-trivial on this subgroup.

Finally, we have the twisted case $D = -4$, in which the field $F_2 = \mathbf{Q}(i)$ having $\varphi_2 = \chi_{-4}$ is different from $K = \mathbf{Q}(\sqrt{r_1})$, and the correction factor in our theorem simply reads

$$E = 1 + E_2 = \begin{cases} 1 & \text{if } f \text{ is odd;} \\ 1 - \chi_{-4}(a) & \text{if } 4|f. \end{cases}$$

If f is odd, the value $A(r, a \bmod f)$ from 5.2 for $\nu(S)$ is correct, and we find $E_2 = 0$ and $E = 1$, just as for 4.2. In the case $4|f$ however, $\varphi_2 = \chi_{-4}$ factors via ρ_2 and the set

$$S_2 = (A_2 \setminus \ker \chi_{-4}) \cap \rho_2^{-1}(a \bmod f\mathbf{Z}_2)$$

does not have the density $\nu_2(S_2) = (2\phi(f_2))^{-1}$ used in 5.2. This is because the congruence set $\rho_2^{-1}(a \bmod f\mathbf{Z}_2)$ is a subset of $A_2 \setminus \ker \chi_{-4}$ if we have $a \equiv -1 \pmod{4}$, and disjoint from it for $a \equiv 1 \pmod{4}$. In this case, $\nu(S)$ is equal to the value $A(r, a \bmod f)$ from 5.2 multiplied by a factor $E = 1 - \chi_{-4}(a) \in \{0, 2\}$. Note that this is a correction for obtaining the true value of $\nu(S)$, not an entanglement correction factor.

The factor $1 - \chi_{-4}(a)$ vanishes for $a \equiv 1 \pmod{4}$, reflecting the fact that if $-r$ is a square, then S_2 is empty as r will be a square modulo any prime congruent to $1 \pmod{4}$, and not a primitive root. In the case $a \equiv -1 \pmod{4}$, when S_2 and therefore S are non-empty, the entanglement correction factor coming out of Theorem 3.3 equals 1 because the average value of the non-cyclotomic character $\chi_2 = \psi_K \cdot \chi_{K,2}$ on the congruence set $S_2 = \rho_2^{-1}(a \bmod f\mathbf{Z}_2)$ equals 0. It follows that in this case, $\delta(S) = \nu(G \cap S)/\nu(G)$ is equal to the naive value

$$E \cdot A(r, a \bmod f) = 2A(r, a \bmod f) = \nu(S). \quad \square$$

As the Artin constant $A(r, a \bmod f)$ from 5.2 is non-zero, vanishing of the density $\delta(S)$ in Theorem 5.3 occurs if and only if the correction factor E vanishes, and $G \cap S$ is empty. It is easy to see when this happens.

5.4. Corollary. *The correction factor E in Theorem 5.3 vanishes if and only if we are in one of the two following cases:*

- (a) *the discriminant of $F_2 = \mathbf{Q}(\sqrt{r})$ divides f , and we have $\chi_{F_2}(a) = 1$;*
- (b) *r is a cube in \mathbf{Q}^* , the discriminant of $\mathbf{Q}(\sqrt{r})$ divides $3f$ but not f , and for $L = \mathbf{Q}(\sqrt{-3r})$ we have $\chi_L(a) = -1$.*

Proof. The factor E in Theorem 5.3 does not vanish if there is a prime $p > 3$ that divides the discriminant D of $F_2 = \mathbf{Q}(\sqrt{r})$ but not f , since then we have $[F_p : \mathbf{Q}] - 1 \geq p - 2 > 1$. This leaves us with two cases in which it does vanish.

The first case arises when all odd primes in D divide f , and we have an equality $E = 1 + E_2 \prod_{p|D \text{ odd}} \left(\frac{a}{p}\right) = 0$. In this case E_2 equals $-\chi_{F_2,2}(a)$, so actually D divides f , and we have $E = 1 - \chi_{F_2}(a) = 0$. This is case (a), in which all primes congruent to $a \bmod f$ are split in $\mathbf{Q}(\sqrt{r})$, making r a quadratic residue modulo all but finitely many of these primes.

The second case arises if all odd primes in D divide f except the prime $p = 3$, which divides D but not f , and we have

$$E = 1 + E_2 \cdot \prod_{p|D/3 \text{ odd}} \left(\frac{a}{p}\right) \cdot \frac{-1}{[F_3 : \mathbf{Q}] - 1} = 0.$$

In this situation E_2 equals $-\chi_{F_2,2}(a)$, so D divides $3f$ but not f , and $[F_3 : \mathbf{Q}]$ equals 2, showing that r is a cube. The resulting equality is $E = 1 + \chi_L(a) = 0$, so we are in case (b). To understand this case, we note that a cube can only be a primitive root modulo a prime $q \equiv 2 \bmod 3$, and no prime q can be inert in all three quadratic subfields of $\mathbf{Q}(\sqrt{r}, \sqrt{-3r})$. \square

Corollary 5.4 already occurs in [3, Theorem 8.3], where it is said to follow from a ‘straightforward analysis’ in terms of Galois groups that is not further specified.

6. NEAR-PRIMITIVE ROOTS

In addition to $r = \pm r_0^e \in \mathbf{Q}^* \setminus \{\pm 1\}$, we now let $t = \prod_p t_p \in \mathbf{Z}_{\geq 1}$ be a positive integer, with $t_p = p^{\text{ord}_p(t)}$ the p -component of p . We are interested in the density of the set of primes q for which r is a ‘near-primitive root’ in the sense that $r \bmod q$ generates a subgroup of exact index t in \mathbf{F}_q^* . Note that in particular, such primes q will be congruent to 1 mod t .

For primes q coprime to $2r$, the condition amounts to requiring that q split completely in $F_t = \mathbf{Q}(R_t)$, but not in any of the fields F_{pt} with p prime. In order to define the set $S \subset A$ associated to this condition, we slightly generalize the condition imposed in 3.2, using the surjective restriction maps

$$(3.1) \quad \varphi_P : A_p \longrightarrow A(P) = \text{Aut}_{R_P \cap \mathbf{Q}^*}(R_P)$$

for p -powers P from Section 3. In this case, we have $S = \prod_p S_p$ for subsets $S_p \subset A_p$ defined by

$$(6.1) \quad S_p = \ker \varphi_{t_p} \setminus \ker \varphi_{pt_p}.$$

Note that 6.1 reduces to 3.2 for $p \nmid t$, when we have $t_p = 1$. In all cases S_p is the difference of a group and a subgroup, just as in Section 4.

In order to compute $\nu(S_p)$, we describe the finite quotients $A(P)$ of A_p in 3.1 much as we described their infinite counterpart $A = \text{Aut}_{R_\infty \cap \mathbf{Q}^*}(R_\infty)$ in Section 2.

Let $r^{1/P} \in \overline{\mathbf{Q}}$ be any root of the polynomial $X^P - r$. Then $R_P = \langle r^{1/P} \rangle \times \mu_P$ is a product of an infinite cyclic group and the finite group μ_P of P -th roots of unity, and its quotient

$$C_P = \frac{R_P}{\mu_P \cdot (R_P \cap \mathbf{Q}^*)}$$

is a finite cyclic group of order dividing P , generated by $r^{1/P} \bmod \mu_P \cdot (R_P \cap \mathbf{Q}^*)$. Just as for A and A_p , we have an exact sequence

$$(6.2) \quad 1 \rightarrow \text{Hom}(C_P, \mu_P) \rightarrow A(P) \rightarrow \text{Aut}(\mu_P) \rightarrow 1.$$

If we are not in the twisted case where $P > 1$ is a 2-power and $-r$ a rational square, then r has a rational (e, P) -th root and C_P is cyclic of order $P/(P, e)$. If $-r = r_0^e$ is a rational square and $P > 1$ a 2-power, there is a slight difference when P divides e . In this case the order of C_P is not $P/(P, e) = 1$ but 2, as $r^{1/P}$ is equal to a primitive $2P$ -th root of unity times the rational number $r_0^{e/P}$. We find that $A(P)$ has order $\phi(P) \cdot P/(P, e)$, unless we are in the twisted case with $P > 1$ a 2-power dividing e , when the order is $2\phi(P) \cdot P/(P, e)$.

Write the exponent e in $r = \pm r_0^e$ as $e = \prod_p e_p$ with $e_p = p^{\text{ord}_p(e)}$. If $p|t$ is a prime number, the set S_p in 6.1 is non-empty, and if we are not in the twisted case $p = 2$ with $-r$ a rational square, its density equals

$$\nu_p(S_p) = \frac{(t_p, e)}{\phi(t_p) \cdot t_p} - \frac{(pt_p, e)}{\phi(pt_p) \cdot pt_p} = \frac{(t_p, e)}{t_p^2} \cdot \begin{cases} 1 & \text{if } pt_p | e_p; \\ 1 + \frac{1}{p} & \text{if } e_p | t_p. \end{cases}$$

If 2 divides t and we are in the twisted case where $-r$ is a square, we need to multiply the value for $\nu(S_2)$ above by a factor

$$\alpha_2 = \begin{cases} 1/2 & \text{if } -r = \square \text{ and } 2t_2 | e_2; \\ 1/3 & \text{if } -r = \square \text{ and } 2|t_2 = e_2; \\ 1 & \text{otherwise.} \end{cases}$$

Taking the product over all p , with α_2 as above, we obtain the analogue

$$(6.3) \quad A(r, t) = \alpha_2 \cdot \frac{(t, e)}{t^2} \cdot \prod_{p|t, e_p | t_p} \left(1 + \frac{1}{p}\right) \cdot \prod_{p \nmid t} \left(1 - \frac{1}{[F_p : \mathbf{Q}]}\right)$$

of the naive Artin constant for primes modulo which r is a near-primitive root generating a subgroup of exact index t .

6.4. Theorem. *Let $r = \pm r_0^e \in \mathbf{Q}^*$ and $t \in \mathbf{Z}_{\geq 1}$ be as above, and define the set $S = \prod_p S_p \subset A$ associated to the primes for which r is a near-primitive root of index t as in 6.1. Then we have*

$$\delta(S) = \frac{\nu(G \cap S)}{\nu(G)} = E \cdot A(r, t)$$

for the Galois group $G \subset A$ from 1.5, with $A(r, t)$ the Artin constant from 6.3 and the correction factor E equal to

$$E = 1 + E_2 \cdot \prod_{p|d, p \nmid 2t} \frac{-1}{[F_p : \mathbf{Q}] - 1}.$$

Here d denotes the discriminant of $K = \mathbf{Q}(\sqrt{r_1})$, with $r_1 = \pm r_0$ defined as in 2.10. The value of E_2 can be given in terms of $d_2 = 2^{\text{ord}_2(d)}$ and

$$s_2 = \begin{cases} \text{lcm}(2e_2, d_2) & \text{if } -r \neq \square; \\ 4 & \text{if } -r = \square, e_2 = 2 \text{ and } d_2 = 8; \\ 4e_2 & \text{otherwise} \end{cases}$$

as

$$E_2 = \begin{cases} 1 & \text{if } s_2 | t_2; \\ 0 & \text{if } s_2 \nmid 2t_2; \\ -1 & \text{if } s_2 = 2t_2 = 2, \text{ or if } \\ & s_2 = 2t_2 = 2e_2 = 4 \text{ with } -r = \square \text{ and } d_2 = 8; \\ -1/3 & \text{otherwise.} \end{cases}$$

Proof. We already computed $\nu(S) = A(r, t)$ in 6.3, so by Theorem 3.3 we only need to check that the correction factor $E = 1 + \prod_{p|2d} E_p$ has the indicated form.

At odd primes $p|d$ outside t , the factor $E_p = -1/([F_p : \mathbf{Q}] - 1)$ is as in 4.1.

At odd primes $p|(d, t)$, the Legendre symbol χ_p equals 1 on $\ker \varphi_{t_p}$, and therefore on S_p . As we have $E_p = 1$ for these p , we obtain

$$E = 1 + E_2 \cdot \prod_{p|d, p \nmid 2t} \frac{-1}{[F_p : \mathbf{Q}] - 1},$$

with E_2 the average value of the character $\chi_2 = \psi_K \cdot \chi_{K,2}$ on the ‘difference of subgroups’ $S_2 = \ker \varphi_{t_2} \setminus \ker \varphi_{2t_2}$. To find E_2 , we need to determine when χ_2 is trivial on $\ker \varphi_{t_2}$ and on $\ker \varphi_{2t_2}$.

For $k \in \mathbf{Z}_{>0}$, the character $\chi_{K,2}$ is trivial on $\ker \varphi_{2^k} \subset A_2$ if and only if its conductor d_2 divides 2^k . For ψ_K , we have

$$\psi_K = 1 \text{ on } \ker \varphi_{2^k} \iff r_1^{1/2} \in R_{2^k} \iff \begin{cases} 2e_2 | 2^k & \text{if } -r \neq \square; \\ 4e_2 | 2^k & \text{if } -r = \square. \end{cases}$$

The case distinction stems from the fact that in the twisted case where $-r$ is a square, R_{2e_2} does not contain $r_1^{1/2}$ itself, but only a product $\zeta_{4e_2} r_1^{1/2}$ with a primitive $4e_2$ -th root of unity. As ζ_{4e_2} is contained in R_{4e_2} , we do have $r_1^{1/2} \in R_{4e_2}$.

The characters ψ_K and $\chi_{K,2}$, which measure the action of automorphisms in A_2 on $r_1^{1/2}$ and the d_2 -th roots of unity, respectively, are defined via the ‘different components’ of A_2 in the sense of 2.5. If we are not in the twisted case, ψ_K and $\chi_{K,2}$ are as unrelated on $\ker \varphi_{2^k}$ as they are on A_2 , and χ_2 is trivial on $\ker \varphi_{2^k}$ if and only if both ψ_K and $\chi_{K,2}$ are. This happens if and only if $\text{lcm}(2e_2, d_2)$ divides 2^k .

Now suppose we are in the twisted case. Then ψ_K is non-trivial on $\ker \varphi_{2e_2}$ as we have $r_1^{1/2} \notin R_{2e_2}$. However, as all $\sigma \in \ker \varphi_{2e_2}$ fix $\zeta_{4e_2} r_1^{1/2}$, it can be described ‘in cyclotomic terms’ as $\psi_K(\sigma) = \zeta_{4e_2}^{\sigma-1}$. Thus, in the case $e_2 = 2$ and $d_2 = 8$, the characters ψ_K and $\chi_{K,2}$ coincide on $\ker \varphi_{2e_2} = \ker \varphi_4$, and their product χ_2 is trivial on it. Apart from this rather special case, χ_2 is trivial on $\ker \varphi_{2^k}$ if and only if $4e_2$ divides 2^k . Note that d_2 , a divisor of 8, necessarily divides $4e_2$. Thus, the 2-power s_2 in the Theorem is defined such that in all cases, χ_2 is trivial on $\ker \varphi_{2^k}$ if and only if s_2 divides 2^k .

If s_2 divides t_2 , then χ_2 is trivial on $\ker \varphi_{t_2}$, hence on S_2 , and we find $E_2 = 1$. If s_2 does not divide $2t_2$, then χ_2 is non-trivial on both $\ker \varphi_{2t_2}$ and $\ker \varphi_{t_2}$, and we find $E_2 = 0$. In the remaining case $s_2 = 2t_2$ we find, just as for 4.1,

$$E_2 = \frac{-\nu_2(\ker \varphi_{2t})}{\nu_2(\ker \varphi_{t_2}) - \nu_2(\ker \varphi_{2t_2})} = \frac{-1}{[A(t_2) : A(2t_2)] - 1}.$$

In the case $s_2 = 2t_2 = 2$, where both e and $d = \text{disc}(\mathbf{Q}(\sqrt{r_1})) = \text{disc}(\mathbf{Q}(\sqrt{r}))$ are odd, the index $[A_2 : A(2)]$ equals 2, and we find $E_2 = -1$. This also happens in the special twisted case above, when we have $s_2 = 2t_2 = 2e_2 = 4$ and $d_2 = 8$; indeed, we then have $[A(2) : A(4)] = 2$ by the order formulas $\#A(2) = 2$ and $\#A(4) = 4$ following 6.2. In the other cases with $s_2 = 2t_2 \geq 4$ we have $e_2 | t_2$, and in the twisted cases with $s_2 = 4e_2$ even $2e_2 | t_2$. The order formulas then yield $\#A(2t_2) = 4 \cdot \#A(t_2)$ and $[A(t_2) : A(2t_2)] = 4$, and we find $E_2 = -1/3$. \square

If we compare 6.4 to the result for near-primitive root densities in [14], we see that, despite the careful administration we needed for the twisted case, both the derivation and the resulting expression for the density given here are considerably simpler. In fact, it takes some work to see that the formulas in [14], which express the density as a sum of up to 4 different inclusion-exclusion-sums, can be reduced to our single formula. Whereas it is extremely cumbersome to derive the vanishing criteria from the formulas in [14], it is straightforward to obtain them from Theorem 6.4. In the criteria below, which occur without proof as [3, (8.9)–(8.13)], we write $d(x)$ for $x \in \mathbf{Z}$ to denote the discriminant of the number field $\mathbf{Q}(\sqrt{x})$. In particular, $d(x)$ equals 1 if x is a square.

6.5. Corollary. *Let $r = \pm r_1^e$ and $t \in \mathbf{Z}_{\geq 1}$ be as in Theorem 6.4. Then the near-primitive root density $E \cdot A(r, t)$ in 6.4 vanishes if and only if we are in one of the following five cases:*

- (a) t is odd, and $d(r)|t$;
- (b) $t \equiv 2 \pmod{4}$, and $r = -u^2$ with $d(2u)|2t$;
- (c) r is a cube, $3 \nmid t$, $-r \neq \square$, $d(-3r_0)|t$, and $\text{ord}_2(t) > \text{ord}_2(e)$;
- (d) r is a cube, $3 \nmid t$, $-r = \square$, $d(-3r_0)|t$ and $\text{ord}_2(t) > \text{ord}_2(e) + 1$;
- (e) r is a cube, $3 \nmid t$, $-r = u^2$, $8|d(-3u)|2t$.

Proof. The naive density $A(r, t)$ vanishes if and only if t is odd and r is a square. This is case (a) with $d(r) = 1$.

As any local factor $E_p = -1/([F_p : \mathbf{Q}] - 1)$ satisfies $|E_p| \leq 1/(p-2) < 1$ for $p \geq 5$, we see that $E = 1 + \prod_p E_p$ can only vanish if we have $E_p = 1$ for all primes $p \geq 5$, i.e., if all primes $p \geq 5$ dividing d also divide t . Assume that this is the case. Then E vanishes if and only if we either have $E_2 = -1 = -E_3$ or $E_2 = 1 = -E_3$.

Suppose first that we have $E = 0$ with $E_2 = -1 = -E_3$. For $s_2 = 2$ this means that t and e and $d = d(r_0) = d(r)$ are odd, and that d divides t . This is case (a) with $d(r) \neq 1$. For $s_2 = 4$ we have $t_2 = 2$ and $r = -u^2$, with $u = r_0^{e/2}$ a non-square rational number for which $d(u) = d(r_0) = d$ satisfies $8|d|4t$. As $8|d(u)$ can be written as $\text{ord}_2(d(2u)) \leq 2 = \text{ord}_2(2t)$, we are in case (b).

Suppose next that we have $E = 0$ with $E_2 = 1 = -E_3$. The condition $E_3 = -1$ means that r is a cube, and that 3 divides d but not t . To have $E_2 = 1$ as well, t_2 needs to be divisible by s_2 , and this leads to three cases reflecting the three cases in the definition of s_2 . In the non-twisted case, t_2 has to be divisible by $2e_2$ and d_2 , leading to $\text{ord}_2(t) > \text{ord}_2(e)$ and $d(-3r_0) = -d(r_0)/3|t$. This is case (c). The twisted case with $s_2 = 4e_2$ is case (d), with $\text{ord}_2(t) > \text{ord}_2(e) + 1$ reflecting the condition $s_2 = 4e_2|t_2$. Finally, we have the twisted case with $s_2 = 4$. Here $-r = u^2$ is a square and $e_2 = 2$, so we have $d = d(u)$ and $d(-3u) = -d(u)/3$. The conditions $4|t$ and $d_2 = 8$ may now be combined with the conditions at the odd primes to yield $8|d(-u/3)|2t$, and we are in case (e).

The reader may check that E indeed vanishes in each of the cases (a)–(e), or refer to remark 6.6.2 below instead. \square

6.6. Remarks. 1. One may restrict case (e) to values $t \equiv 4 \pmod{8}$, as $t \equiv 0 \pmod{8}$ in case (e) is already covered by case (d). In doing so, the five cases become mutually exclusive.

2. The computation of the vanishing criteria in 6.5 is so automatic that one barely realizes *why* these are vanishing criteria. In case (a) the number r is a square modulo almost all primes $q \equiv 1 \pmod{t}$, so it cannot generate a subgroup of odd index modulo such q for $q > 2$. In case (b), if $r = -u^2$ generates a subgroup of even index modulo q , then $(\frac{r}{q}) = (\frac{-1}{q}) = 1$ implies that we have $q \equiv 1 \pmod{4}$, and $r = (iu)^2 \pmod{q}$ for a primitive 4-th root of unity i modulo q . For $q \equiv 1 \pmod{t}$ we easily see that q splits in $\mathbf{Q}(\sqrt{u})$ if and only if we have $q \equiv 1 \pmod{8}$, so iu is a square modulo q and r generates a subgroup of index divisible by 4 modulo q .

In the cases (c)–(e), the divisibility of the index of r modulo q by t implies that -3 is a square modulo q , so we have $q \equiv 1 \pmod{3}$, and the cube r generates a subgroup of index divisible by 3.

7. GENERALIZATIONS

The examples in the two preceding sections show that the character sum approach to the computation of various primitive root densities gives rise to formulas with a simple basic structure. Unsurprisingly, more case distinctions become necessary if the complexity of the input data grows. In more complicated settings, where a single closed formula running over a page of case distinctions may not be the most desirable result, the method can also be seen as an *algorithm* to find the density in each specific case.

Near-primitive roots for primes in arithmetic progressions. As a rather straightforward generalization, one may combine Sections 5 and 6 into a single density computation for the set of primes $q \equiv a \pmod{f}$ for which a given rational number $r = \pm r_0^e$ generates a subgroup of exact index t in \mathbf{F}_q^* . As such primes q are necessarily congruent to 1 mod t , it is natural to assume $t|f$ and $a \equiv 1 \pmod{t}$. For primes $p|f$, the original Artin sets 3.2 are then replaced in the spirit of 5.1 and 6.1 by

$$S_p = (\ker \varphi_{t_p} \setminus \ker \varphi_{pt_p}) \cap \rho_p^{-1}(a \pmod{f\mathbf{Z}_p}).$$

One can now compute the values $\nu_p(S_p)$ and their somewhat complicated product $A(r, t, a \pmod{f})$ over all p as before. Application of Theorem 3.3 yields the fraction $\delta(S) = \nu(G \cap S)/\nu(G)$ as a product of $A(r, t, a \pmod{f})$ and a correction factor $E = 1 + \prod_p E_p$, where the value of E_2 requires a large number of case distinctions. We leave the details to the reader fond of general closed formulas, and note that when viewed as an *algorithm*, the method easily yields $\delta(S)$ for any set of (factored) input values t, f, r from \mathbf{Z} and \mathbf{Q}^* .

Two-variable Artin conjectures. There are generalizations of Theorem 3.3 to variants of Artin's conjecture over \mathbf{Q} for which not the theorem itself, but the method of proof applies with few changes. One might for instance want to compute, upon input of $r, s \in \mathbf{Q}^*$, the density of primes q for which \mathbf{F}_q^* is generated by r and s , or for which s is in the subgroup of \mathbf{F}_q^* generated by r . We assume we are in the true 2-variable case where r and s are multiplicatively independent, i.e., when they generate a subgroup of rank 2 in $\mathbf{Q}^*/\{\pm 1\}$.

In this case, we are led to study the Galois group G of the extension $\mathbf{Q} \subset \mathbf{Q}(R_\infty, S_\infty)$ obtained by adjoining to \mathbf{Q} all radicals of r and all radicals of s . Analogous to 1.5, one has the injective Galois representation

$$G \longrightarrow A = \text{Aut}_{R_\infty \cdot S_\infty \cap \mathbf{Q}^*}(R_\infty \cdot S_\infty).$$

The group A is an extension of $\text{Aut}(\mu_\infty) = \widehat{\mathbf{Z}}^*$ by a free $\widehat{\mathbf{Z}}$ -module of rank 2 that naturally decomposes as a product $A = \prod_p A_p$ of automorphism groups of p -power radicals. The direct analogue of Theorem 2.7 is that $G \subset A$ is a subgroup of index four that arises as the intersection of the kernels of *two* quadratic characters $\kappa = \psi_K \cdot \chi_K$ and $\kappa' = \psi_{K'} \cdot \chi_{K'}$ on A related to entanglement fields $K = \mathbf{Q}(\sqrt{r_0})$ and $K' = \mathbf{Q}(\sqrt{r'_0})$. Here $r_0, r'_0 \in \mathbf{Q}^*$ are rational numbers that lift a \mathbf{Z} -basis of the rank-2 free abelian subgroup $(R_\infty \cdot S_\infty \cap \mathbf{Q}^*)/\{\pm 1\} \subset \mathbf{Q}^*/\{\pm 1\}$ to \mathbf{Q}^* .

For subsets $S = \prod_p S_p \subset \prod_p A_p = A$, the analogue of Theorem 3.3 is that the quotient $\nu(G \cap S)/\nu(G)$ differs from $\nu(S)/\nu(A)$ by an entanglement correction factor of the form

$$1 + \prod_p E_{\kappa,p} + \prod_p E_{\kappa',p} + \prod_p E_{\kappa\kappa',p},$$

with $E_{\alpha,p}$ denoting, for a character $\alpha = \prod_p \alpha_p$ on $A = \prod_p A_p$, the average value of α_p on S_p . It reflects the fact that in this case, $\frac{1}{4}(1+\kappa+\kappa'+\kappa\kappa')$ is the characteristic function of G in A . This leads to much easier proofs of results such as [8, Theorem 3].

Further generalizations. Nothing prevents us from considering properties of subgroups of \mathbf{F}_q^* that are generated by n elements $a_1, a_2, \dots, a_n \in \mathbf{Q}^*$ for any $n \in \mathbf{Z}_{>0}$. One may for instance look at those q for which *all* a_i are primitive roots modulo q , or those q for which the subgroup $\Gamma = \langle a_1, a_2, \dots, a_n \rangle \subset \mathbf{Q}^*$ maps surjectively to \mathbf{F}_q^* . Our methods do generalize to this situation, and lead to proofs of theorems obtained previously by Matthews [5] and Cangelmi and Pappalardi [1].

The ultimate structural result on the Galois group G over \mathbf{Q} of the field obtained by adjoining to \mathbf{Q} the group $R = \sqrt[n]{\mathbf{Q}}$ of all radicals of *all* rational numbers is that G is the subgroup of $A = \text{Aut}_{R \cap \mathbf{Q}^*}(R)$ that is ‘cut out’ by an explicit family of quadratic characters. It consists, for each prime p , of a character as in 2.7 that expresses the fact that $\sqrt[p]{p}$ equals a ‘Gauss sum’, a sum of roots of unity, and that elements of G should preserve this *additive* relation. It implies that over \mathbf{Q} , the groups of radical Galois extensions $\mathbf{Q} \subset \mathbf{Q}(W)$ for subgroups $W \subset R$ can be described as subgroups of the automorphism group $\text{Aut}_{\mathbf{Q}^* \cap W}(W)$ that arise as the intersections of kernels of certain quadratic characters.

A beautiful generalization of this result to arbitrary number fields K was announced in the 2006 lecture notes [4, Section 13] of the first author. It describes the Galois group over K of the maximal radical extension $K(\sqrt[n]{K})$ of K explicitly as a subgroup G of $A = \text{Aut}_{K^* \cap \sqrt[n]{K}}(\sqrt[n]{K})$. In all cases, G is normal in A , and A/G is a profinite *abelian* group. It opens up the possibility of generalizing all results that have been proved or mentioned over \mathbf{Q} in this paper to similar results over arbitrary number fields. Such extensions, and also generalizations that replace the multiplicative group by one-dimensional tori, are the subject of ongoing work of De Smit and Palenstijn [12].

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